

Affine Gravity, Palatini Formalism and Charges^{*}

Joseph Katz^{1†} and Gideon I. Livshits^{2‡}

¹*The Racah Institute of Physics*

²*Institute of Chemistry*

The Hebrew University, Givat Ram, 91904 Jerusalem, Israel

August 5, 2011

Abstract

Affine gravity and the Palatini formalism contribute both to produce a simple and unique formula for calculating charges at spatial and null infinity for Lovelock type Lagrangians whose variational derivatives do not depend on second-order derivatives of the field components. The method is based on the covariant generalization due to Julia and Silva of the Regge-Teitelboim procedure that was used to define properly the mass in the classical formulation of Einstein's theory of gravity. Numerous applications reproduce standard results obtained by other secure but mostly specialized methods. As a novel application we calculate the Bondi energy loss in five dimensional gravity, based on the asymptotic solution given by Tanabe, Tanahashi and Shiromizu, and obtain, as expected, the same result. We also give the superpotential for Einstein-Gauss-Bonnet gravity and find the superpotential for Lovelock theories of gravity when the number of dimensions tends to infinity with maximally symmetrical boundaries.

The paper is written in standard component formalism.

Keywords: affine gravity, Palatini formalism, Lovelock Gravity, charges at null infinity.

^{*}The work is dedicated to Joshua Goldberg from whom I learned and got interested in conservation laws in General Relativity (J.K).

[†]email: jkatz@phys.huji.ac.il

[‡]email: livshits.gideon@mail.huji.ac.il

1 Introduction

(i) Hamiltonian formalism and the conservation of charges

The way to evaluate total energy in the Hamiltonian formulation of general relativity [1] was put on a sound basis by Regge and Teitelboim [29] for localized sources¹ in spacetimes that are asymptotically flat. The correct Hamiltonian H has the following particularities: (1) The field equations are derived from a volume integral generator. (2) The generator is zero as a consequence of the equations, i.e. on-shell. (3) For any solution of Einstein's equations with given boundary conditions, H reduces to a surface integral on the boundary and with the boundary at spatial infinity the Hamiltonian is equal to the total mass-energy of the sources E including gravitational energy².

In the covariant Hamiltonian formalism or covariant phase space as described by Julia and Silva³ [19] the Hamiltonian appears in a slightly different light: (1) H is now associated with a timelike global vector field ξ , and is an integral over a spacelike hypersurface of a vector density⁴ $\hat{\mathbf{J}}$. (2) The generator of the field equations is an integral over a spacelike hypersurface of a vector density current $\hat{\mathbf{W}}$, a linear combination of variational derivatives which is thus equal to zero on-shell. (3) The Hamiltonian on-shell reduces to a surface integral on the boundary S of the domain of integration V . The surface integral is generated by an anti-symmetric tensor density $\hat{\mathbf{U}}$ defined on the boundary; this is the superpotential. The surface integral is equal to the energy E provided the timelike vector field becomes pure time translations at spatial flat infinity.

We have thus the following sequence of equalities⁵ in covariant phase space:

$$\begin{aligned} \hat{J}^\mu &= \hat{W}^\mu + \partial_\nu \hat{U}^{\mu\nu} \quad \Rightarrow \quad H = \int_V \hat{J}^\mu dV_\mu = \int_V \hat{W}^\mu dV_\mu + \oint_S \hat{U}^{\mu\nu} dS_{\mu\nu}; \\ \text{on shell } \hat{W}^\mu &= 0 \quad , \quad \partial_\mu \hat{J}^\mu = 0; \quad \text{and} \quad H = E = \oint_S \hat{U}^{\mu\nu} dS_{\mu\nu}. \end{aligned} \quad (1.1)$$

In both cases E is the time component of a 4-vector in Minkowski spacetime⁶. If the timelike vector field ξ is replaced by a spacelike vector field which turns (fast enough) into space translations along one of the spatial axes at infinity, the surface integral will be a

¹Localized matter or fields that decrease faster than $1/r$ at flat spatial infinity.

² The method has been used in a variety of contexts, for a relatively recent application of the Regge-Teitelboim method see, for instance, Jamsin [14].

³See also [18] which contains most references to original works on the subject.

⁴The density character is indicated by a hat “ $\hat{}$ ” over a symbol.

⁵Greek indices go from 0 to $N-1$, N is the dimension of spacetime with signature $-(N-2)$. $N \geq 4$.

⁶Various examples of applications can be found in [18].

component of the linear momentum along that axis. The way to generate angular momentum components is now obvious.

The formalism holds for fields other than gravity with other global or asymptotic symmetries which are associated with other constants of motion or conserved quantities or what are generally called charges⁷. The whole machinery works also on a hypersurface at null infinity and allows one to calculate time-dependent quantities of physical relevance like the Bondi mass [5], the Sachs linear momentum [31] and the angular-momentum [21] with its usual ambiguities in four dimensions but apparently with no ambiguity in 5 dimensions [34].

(ii) The superpotential

The superpotential $\hat{\mathbf{U}}$ generates thus the conserved or non-conserved charges associated with the symmetries or Killing vectors of the metric on the boundary.

There are two problems with the superpotential.

(1) It must be calculated in Minkowski coordinates⁸ in spite of the fact that spherical coordinates are more convenient on a sphere, but in other coordinates the surface integral most often diverges. To avoid the latter one needs to introduce a background [20], [23] or counter-terms [11], [27] or use a regularization procedure [24] or some other device [2].

(2) More importantly, the superpotential is not unique because the Lagrangian is not unique.

We shall remove the coordinate difficulty by introducing a background metric $\bar{\mathbf{g}}$. The utility of backgrounds was suggested by Rosen [30] in the 1940's and often used since then in various publications, but we shall not take it as far as Rosen himself. The background metric $\bar{\mathbf{g}}$ is defined as follows: take the metric \mathbf{g} to be a solution of Einstein's equations that satisfies the boundary conditions and remove the source terms from the solution. Make them equal to zero. The result must be the metric far from the sources and thus on or near the boundary (at spatial infinity) $\mathbf{g} = \bar{\mathbf{g}}$. Notice that the definition of the background is perfectly covariant. We shall never use the background as a "background" metric anywhere else but on the boundary⁹, so we shall not have to define a mapping of our spacetime onto the

⁷The word charge without qualification may have different meaning for different authors. Here we used the word charge as it is understood, for instance, in Hollands, Ishibashi and Marolf [11] as constants of motions, or non constant if at null infinity, associated with asymptotic (and global) symmetries. Iyer and Wald [13] define Noether charges. These are not quite the same and are more like some generalizations of Komar's covariant charges.

⁸See for instance Goldberg's [10] calculation of the Bondi Mass.

⁹In some cases, in particular in cosmological spacetimes, a background turns out to be useful [4].

background. If spacetime is asymptotically flat, $\bar{\mathbf{g}}$ is the Minkowski metric in any appropriate coordinates used for the solution of Einstein's equations. We then define the superpotential with respect to that background: $\hat{\mathbf{U}}(\mathbf{g}) - \hat{\mathbf{U}}(\bar{\mathbf{g}})$. This is a covariant tensor which reduces to $\hat{\mathbf{U}}(\mathbf{g})$ in Minkowski coordinates because in those coordinates $\hat{\mathbf{U}}(\bar{\mathbf{g}}) = 0$. The background solves the coordinate problem and allows us to calculate the total energy E in arbitrary coordinates at infinity, most usually in spherical coordinates.

The superpotential non-uniqueness due to the non-uniqueness of the Lagrangian density is a different problem. It is best resolved by the method suggested by Regge and Teitelboim¹⁰ [29]. The *variation* of the Hamiltonian should have no boundary terms by analogy with classical mechanics. Thus the Hamiltonian H should contain a boundary term $\oint \hat{\mathbf{U}} d\mathbf{S}$ but its variation δH should not. This will define $\delta \hat{\mathbf{U}}(\mathbf{g})$. Since the effective Hamiltonian generator is a linear combination of variational derivatives, $\delta \hat{\mathbf{U}}(\mathbf{g})$ thus defined is independent of any divergence added to the Lagrangian. Hence the superpotential is defined on the boundary up to an integration “constant”. A complete unique superpotential is then obtained by taking the integration “constant” equal to $-\hat{\mathbf{U}}(\bar{\mathbf{g}})$.

(iii) Superpotentials, Affine Gravity and the Palatini formalism

Silva [32] generalized the Regge-Teitelboim idea about the superpotential to the covariant Hamiltonian formulation of a large class of field theories, those in which the Lagrangians *and* their variational derivatives depend at most on first-order derivatives of the field components; moreover, the Lie derivatives of the fields with respect to a vector field ξ depend at most on first-order derivatives of ξ as well. Therefore, application to metric theories of gravity was done in affine- $GL(N; \mathbb{R})$ gravity. The more familiar and practical Palatini formalism would not do because the Lie derivatives of the connection coefficients, the Γ 's, contain second order-derivatives of ξ .

The Silva covariant generalization of the Regge-Teitelboim method as well as subsequent developments [19] and applications were presented in terms of forms, within the “covariant symplectic” formulation. Our first intention here is to develop the formalism in component notations.

Affine gravity introduces an extra set of N vectors¹¹ θ^a which constitute a local reference frame at each point of spacetime. This frame is arbitrary and the Lagrangian is invariant

¹⁰See also Henneaux and Teitelboim [12] for anti-de Sitter backgrounds.

¹¹Latin indices which enumerate the vectors vary also from 0 to $N - 1$.

under any linear transformations $\theta^a(x) = \Lambda^a_b(x)\theta^b(x)$. One motivation for this detour was to ensure that all Lie derivatives depend at most on first-order derivatives of ξ . Affine gravity is however not necessary to obtain the same results [22]. Therefore in a second step we shall review how to get the superpotential directly in the Palatini formalism and in component notations, in affine gravity and in the Palatini formulation.

The formula for the superpotential we arrive at uses both affine gravity and Palatini formalism and as far as we are aware of is new. We also give various applications, some of which are new.

(iv) Comment

We want to emphasize that the superpotential exists only on the boundaries, which may be at spatial infinity or at null infinity. The closed boundary may be anywhere but in practice it should always be taken far away from the sources of gravity otherwise the value of the Hamiltonian is meaningless because gravitational energy is not localizable.

2 Lagrangian variations in Palatini variables and in affine gravity

In the Palatini formulation of gravity theories the fields are the metric of spacetime \mathbf{g} with components $g_{\mu\nu}$ and the symmetric connection¹² Γ with components $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$; these fields are independent. The curvature tensor does not have all the symmetries of the Riemann tensor, in particular $\partial_\mu \Gamma^\lambda_{\nu\lambda} \neq \partial_\nu \Gamma^\lambda_{\mu\lambda}$. To avoid any confusion we shall denote the curvature tensor, like in Eisenhart [8], by $B^\lambda_{\nu\rho\sigma}$ and keep the usual notation of the Riemann tensor $R^\lambda_{\nu\rho\sigma}$ for when the connection coefficients are Christoffel symbols. $B^\lambda_{\nu\rho\sigma}$ is antisymmetrical in the last two indices only, $B^\lambda_{\nu\rho\sigma} = -B^\lambda_{\nu\sigma\rho}$. Additional symmetry properties, similar to those of the Riemann curvature tensor, are:

$$B^\lambda_{(\nu\rho\sigma)} = 0, \quad B^\lambda_{\nu(\rho\sigma;\mu)} = 0 \quad \text{and} \quad D_\lambda B^\lambda_{\nu\rho\sigma} = D_\rho B_{\nu\sigma} - D_\sigma B_{\nu\rho} \quad \text{where} \quad B_{\sigma\nu} = B^\lambda_{\sigma\lambda\nu} \neq B_{\nu\sigma}. \quad (2.1)$$

An occasional semi-column represents a covariant differentiation. The curvature tensor itself $B^\lambda_{\nu\rho\sigma}$ is given by

$$B^\lambda_{\nu\rho\sigma} = 2(\partial_{[\rho}\Gamma^\lambda_{\sigma]\nu} + \Gamma^\lambda_{\mu[\rho}\Gamma^\mu_{\sigma]\nu}). \quad (2.2)$$

¹²Non symmetric connections are often considered also.

Now consider a Lagrangian scalar density $\hat{\mathcal{L}}$ which depends on the metric components and the connection coefficients

$$\hat{\mathcal{L}} = \sqrt{-g} \mathcal{L}(g^{\mu\nu}, B^\lambda_{\nu\rho\sigma}). \quad (2.3)$$

\mathcal{L} is a scalar and a hat over a symbol means it is multiplied by $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$. The matter Lagrangian for isolated systems is not necessary in what follows. This implies that we take the boundary of spacetime outside of the matter, in practice at spatial or null infinity, or that fields tend to infinity faster than $1/r$.

Field equations are the variational derivatives equated to zero. The variation of $\hat{\mathcal{L}}$ for arbitrary variations of the field components $\delta g^{\mu\nu}$ and $\delta \Gamma^\lambda_{\mu\nu}$ may thus be written in the following form

$$\delta \hat{\mathcal{L}} = * \hat{\mathcal{L}}_{\mu\nu} \delta g^{\mu\nu} + \hat{\mathcal{L}}^\mu_{\lambda} \delta \Gamma^\lambda_{\mu\nu} + \partial_\mu \hat{\mathcal{E}}^\mu. \quad (2.4)$$

The content of $\hat{\mathcal{E}}^\mu$ depends on the Lagrangian density which is defined up to a divergence and will not concern us here. The variational derivatives are

$$* \hat{\mathcal{L}}_{\mu\nu} = \frac{\partial \hat{\mathcal{L}}}{\partial g^{\mu\nu}} \quad \text{and} \quad \hat{\mathcal{L}}^\mu_{\lambda} = \frac{\delta \hat{\mathcal{L}}}{\delta \Gamma^\lambda_{\mu\nu}}; \quad (2.5)$$

∂ stands for partial derivatives and δ for variational derivatives.

In affine metric-theories, the central element of the description is a set of N^2 vector field components θ^a_μ associated with a local basis in the local tangent spaces. There are $N(N+1)/2$ scalar products denoted by γ^{ab} :

$$\gamma^{ab} = g^{\mu\nu} \theta^a_\mu \theta^b_\nu. \quad (2.6)$$

The symmetric connection Γ is replaced by a spin connection ω which is a vector:

$$\omega^a_{\mu b} = \theta^a_\lambda (\partial_\mu \theta_b^\lambda + \Gamma^\lambda_{\mu\nu} \theta_b^\nu) = \theta^a_\nu D_\mu \theta_b^\nu, \quad (2.7)$$

θ_b^ν is the inverse matrix of θ^a_ν . Reciprocally,

$$\Gamma^\lambda_{\mu\nu} = \theta^b_\mu (\theta_a^\lambda \omega^a_{\nu b} - \partial_\nu \theta_b^\lambda). \quad (2.8)$$

Since Γ is symmetrical in $\mu\nu$, the anti-symmetric part of the right hand side is of course equal to zero. In terms of θ and ω , the curvature tensor becomes

$$B^\lambda_{\nu\rho\sigma} = 2\theta_a^\lambda \theta^b_\nu (\partial_{[\rho} \omega^a_{\sigma]b} + \omega^a_{[\rho c} \omega^c_{\sigma]b}) = \theta_a^\lambda \theta^b_\nu B^a_{b\rho\sigma}. \quad (2.9)$$

$B^a_{b\rho\sigma}$ depends only on ω and its first-order derivatives.

The Lagrangian density $\hat{\mathcal{L}}$, a functional of \mathbf{g} and $\mathbf{\Gamma}$, see (2.2) and (2.3), may be expressed in terms θ^a, γ^{ab} and ω^a_b . However, it will make things somewhat simpler¹³, up to a point, in what follows if instead of θ^a, γ^{ab} and ω^a_b we use θ^a, \mathbf{g} and ω^a_b . It is thus understood that

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}(g^{\mu\nu}, \Gamma^\lambda_{\mu\nu}) = \hat{\mathcal{L}}(g^{\mu\nu}, \theta^a_\mu, \omega^a_{\mu b}). \quad (2.10)$$

The variations of the Lagrangian will be written as usual in terms of variational derivatives which provide the field equations. Thus

$$\delta \hat{\mathcal{L}} = * \hat{\mathcal{L}}_{\mu\nu} \delta g^{\mu\nu} + \bullet \hat{\mathcal{L}}_a^\mu \delta \theta^a_\mu + \hat{\mathcal{L}}_a^{\mu b} \delta \omega^a_{\mu b} + \partial_\mu \hat{\mathcal{D}}^\mu, \quad (2.11)$$

in which

$$* \hat{\mathcal{L}}_{\mu\nu} = \frac{\partial \hat{\mathcal{L}}}{\partial g^{\mu\nu}} \quad , \quad \bullet \hat{\mathcal{L}}_a^\mu = \frac{\partial \hat{\mathcal{L}}}{\partial \theta^a_\mu} \quad , \quad \hat{\mathcal{L}}_a^{\mu b} = \frac{\partial \hat{\mathcal{L}}}{\partial \omega^a_{\mu b}}. \quad (2.12)$$

The explicit form of $\hat{\mathcal{D}}^\mu$, like that of $\hat{\mathcal{E}}^\mu$, is not important.

We shall now go back from (2.11) to (2.4). To this effect we first replace $\delta \omega^a_{\mu b}$ in (2.11) by its variations deduced from (2.7) in terms of $\delta \theta$ and $\delta \Gamma$:

$$\delta \omega^a_{\mu b} = \delta \theta^a_\nu D_\mu \theta_b^\nu + \theta^a_\nu D_\mu \delta \theta_b^\nu + \theta^a_\lambda \theta_b^\nu \delta \Gamma^\lambda_{\mu\nu}. \quad (2.13)$$

The third term on the right hand side of (2.11) becomes, with minor obvious derivations by part in the square brackets¹⁴:

$$\begin{aligned} \hat{\mathcal{L}}_a^{\mu b} \delta \omega^a_{\mu b} = & \left(\hat{\mathcal{L}}_a^{\mu b} D_\mu \theta_b^\nu \right) \delta \theta^a_\nu + \left[\partial_\mu \left(\theta^a_\nu \hat{\mathcal{L}}_a^{\mu b} \delta \theta_b^\nu \right) - D_\mu \theta^a_\nu \hat{\mathcal{L}}_a^{\mu b} \delta \theta_b^\nu - \theta^a_\nu \partial_\mu \hat{\mathcal{L}}_a^{\mu b} \delta \theta_b^\nu \right] \\ & + \theta^a_\lambda \hat{\mathcal{L}}_a^{\mu b} \theta_b^\nu \delta \Gamma^\lambda_{\mu\nu}. \end{aligned} \quad (2.14)$$

This formula will take a nicer form if we introduce the following self explanatory new notations:

$$\theta^a_\lambda \hat{\mathcal{L}}_a^{\mu b} = \hat{\mathcal{L}}_\lambda^{\mu b}; \quad \text{similarly} \quad \hat{\mathcal{L}}_\lambda^{\mu b} \theta_b^\nu = \hat{\mathcal{L}}_\lambda^{\mu\nu} \quad , \quad \theta^a_\lambda \bullet \hat{\mathcal{L}}_a^\mu = \bullet \hat{\mathcal{L}}_\lambda^\mu \quad \text{and} \quad g^{\mu\nu} * \hat{\mathcal{L}}_{\nu\lambda} = * \hat{\mathcal{L}}_\lambda^\mu. \quad (2.15)$$

With these new notations we may rewrite (2.14) as follows

$$\hat{\mathcal{L}}_a^{\mu b} \delta \omega^a_{\mu b} = D_\nu \hat{\mathcal{L}}_\lambda^{\mu\nu} \theta_a^\lambda \delta \theta^a_\mu + \hat{\mathcal{L}}_\lambda^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} + \partial_\nu \left(\hat{\mathcal{L}}_\lambda^{\mu\nu} \theta_a^\lambda \delta \theta^a_\mu \right). \quad (2.16)$$

Inserting (2.16) back into (2.11), we find that

$$\delta \hat{\mathcal{L}} = * \hat{\mathcal{L}}_{\mu\nu} \delta g^{\mu\nu} + \hat{\mathcal{L}}_\lambda^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} + \left(\bullet \hat{\mathcal{L}}_a^\mu + \theta_a^\lambda D_\nu \hat{\mathcal{L}}_\lambda^{\mu\nu} \right) \delta \theta^a_\mu + \partial_\mu \hat{\mathcal{E}}^\mu. \quad (2.17)$$

¹³Though not in the spirit of affine gravity theory!

¹⁴Partial derivatives equal covariant derivatives of scalars, vector densities and anti-symmetric tensor densities.

(2.17) looks like (2.4) except that in the Palatini formulation, there is no θ -dependence¹⁵. Therefore, the factor of $\delta\theta^a_\mu$ must be identically zero:

$$\bullet \hat{\mathcal{L}}_a^\mu + \theta_a^\lambda D_\nu \hat{\mathcal{L}}_\lambda^{\mu\nu} = 0 \quad \text{or} \quad \bullet \hat{\mathcal{L}}_\lambda^\mu + D_\nu \hat{\mathcal{L}}_\lambda^{\mu\nu} = 0. \quad (2.18)$$

This identity will be valuable later on: it connects a variational derivative in affine gravity with the divergence of a variational derivative in the Palatini formalism. Equation (2.17) reduces thus to

$$\delta \hat{\mathcal{L}} = * \hat{\mathcal{L}}_{\mu\nu} \delta g^{\mu\nu} + \hat{\mathcal{L}}_\lambda^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda + \partial_\mu \hat{\mathcal{E}}^\mu, \quad (2.19)$$

which is the same as (2.4). We notice that $*\hat{\mathcal{L}}_{\mu\nu}$ and $\hat{\mathcal{L}}_\lambda^{\mu\nu} = \theta^b_\lambda \theta_a^\mu \hat{\mathcal{L}}_b^{a\nu}$ depend only on \mathbf{g} and \mathbf{B} . Affine gravity and the Palatini formalism of Lagrangians that depend only on the metric and on the curvature tensor but not on their derivatives lead to *equivalent* equations as expected. The variational derivatives are *identical* when the vector fields θ are in the directions of the local axis and $\theta^a_\mu = \delta^a_\mu$.

3 Symmetries and conserved currents in affine gravity

(i) Diffeomorphism invariance in affine gravity

Conservation laws related to the arbitrariness of the coordinates are normally derived following Noether's procedure. One considers the Lie derivative of the Lagrangian density $\delta_L \hat{\mathcal{L}}$ with respect to an arbitrary vector field ξ . The Lie derivative of a scalar density is a divergence $\delta_L \hat{\mathcal{L}} = \partial_\mu (\hat{\mathcal{L}} \xi^\mu)$. In (2.11) we replace the variations of the field components $\delta g^{\mu\nu}$, $\delta \theta^a_\mu$ and $\delta \omega^a_{\mu b}$ by their Lie derivatives as well. These do not contain higher than first-order derivatives of ξ :

$$\begin{aligned} \delta_L g^{\mu\nu} &= (D_\lambda g^{\mu\nu}) \xi^\lambda - 2g^{\lambda(\mu} D_\lambda \xi^{\nu)} \quad , \quad \delta_L \theta^a_\mu = (D_\lambda \theta^a_\mu) \xi^\lambda + \theta^a_\lambda D_\mu \xi^\lambda, \\ \delta_L \omega^a_{\mu b} &= (D_\lambda \omega^a_{\mu b}) \xi^\lambda + \omega^a_{\lambda b} D_\mu \xi^\lambda. \end{aligned} \quad (3.1)$$

We may thus rewrite (2.11) with Lie derivatives as follows:

$$* \hat{\mathcal{L}}_{\mu\nu} \delta_L g^{\mu\nu} + \bullet \hat{\mathcal{L}}_a^\mu \delta_L \theta^a_\mu + \hat{\mathcal{L}}_a^{\mu b} \delta_L \omega^a_{\mu b} = \partial_\nu (\hat{\mathcal{L}} \xi^\nu - \hat{\mathcal{D}}^\nu). \quad (3.2)$$

¹⁵ Suppose we take variations of the fields that are all zero on the boundary and integrate over spacetime. The volume integral should not contain any $\delta\theta$ terms since $\delta\theta$ is arbitrary.

From (3.1) it is easily seen that the divergence is of a vector density linear in ξ and $D\xi$, say of the form

$$\hat{\mathcal{L}}\xi^\nu - \hat{\mathcal{D}}^\nu = *j_\lambda^\nu \xi^\lambda + *\hat{U}_\lambda^{\nu\mu} D_\mu \xi^\lambda. \quad (3.3)$$

The left-hand side is equally linear in ξ and $D\xi$. Thus (3.2) may be rewritten as follows:

$$\hat{\Lambda}_\lambda \xi^\lambda + \hat{\Lambda}^\mu{}_\lambda D_\mu \xi^\lambda = \partial_\nu \left(*j_\lambda^\nu \xi^\lambda + *\hat{U}_\lambda^{\nu\mu} D_\mu \xi^\lambda \right), \quad (3.4)$$

in which

$$\hat{\Lambda}_\lambda = *\hat{\mathcal{L}}_{\mu\nu} D_\lambda g^{\mu\nu} + \bullet\hat{\mathcal{L}}_a^\mu D_\lambda \theta_\mu^a + \hat{\mathcal{L}}_a^{\mu b} D_\lambda \omega_{\mu b}^a, \quad \hat{\Lambda}^\mu{}_\lambda = -2*\hat{\mathcal{L}}_\lambda^\mu + \bullet\hat{\mathcal{L}}_\lambda^\mu + \hat{\mathcal{L}}_a^{\mu b} \omega_{\lambda b}^a. \quad (3.5)$$

It is interesting to notice and easy to verify that $-2*\hat{\mathcal{L}}_\lambda^\mu + \bullet\hat{\mathcal{L}}_\lambda^\mu$ is, in affine gravity, the variational derivative with respect to θ_μ^a times θ_λ^a :

$$\hat{\mathcal{L}}_\lambda^\mu = -2*\hat{\mathcal{L}}_\lambda^\mu + \bullet\hat{\mathcal{L}}_\lambda^\mu = \left. \frac{\partial \hat{\mathcal{L}}}{\partial \theta_\mu^a} \theta_\lambda^a \right|_{\gamma, \omega}. \quad (3.6)$$

Now in (3.4) replace¹⁶ $\xi(x)$ by $\xi(x)\epsilon(x)$; this can always be done because ξ is arbitrary. Then expand in terms of derivatives of $\epsilon(x)$. The left-hand side of (3.4) looks then like this:

$$(\hat{\Lambda}_\lambda \xi^\lambda + \hat{\Lambda}^\mu{}_\lambda D_\mu \xi^\lambda) \epsilon + (\hat{\Lambda}^\mu{}_\lambda \xi^\lambda) D_\mu \epsilon. \quad (3.7)$$

On the other hand, the right hand side is now of the following *form*:

$$\partial_\nu (*j^\nu \epsilon + *\hat{U}^{\nu\mu} \partial_\mu \epsilon) = (\partial_\nu *j^\nu) \epsilon + (*j^\mu + \partial_\nu *\hat{U}^{\nu\mu}) \partial_\mu \epsilon + (*\hat{U}^{\nu\mu}) D_{\mu\nu} \epsilon. \quad (3.8)$$

Thus, since (3.7) and (3.8) are equal

$$(\hat{\Lambda}_\lambda \xi^\lambda + \hat{\Lambda}^\mu{}_\lambda D_\mu \xi^\lambda) \epsilon + (\hat{\Lambda}^\mu{}_\lambda \xi^\lambda) D_\mu \epsilon = (\partial_\nu *j^\nu) \epsilon + (*j^\mu + \partial_\nu *\hat{U}^{\nu\mu}) \partial_\mu \epsilon + (*\hat{U}^{\nu\mu}) D_{\mu\nu} \epsilon. \quad (3.9)$$

ϵ is arbitrary and we may identify the factors of ϵ , $\partial_\mu \epsilon$ and of $D_{\mu\nu} \epsilon$ of both sides. There follows a set of “cascade identities” [16]. The factors of ϵ reproduce of course identity (3.4). The factor of $D_{\mu\nu} \epsilon$ does not appear on the left-hand side ; thus $*\hat{U}^{\nu\mu}$ must be antisymmetric:

$$*\hat{U}^{\nu\mu} = -*\hat{U}^{\mu\nu}. \quad (3.10)$$

Taking this into account, the identification of the factors of $\partial_\mu \epsilon$ leads to the following identity:

$$*j^\mu = *\hat{W}^\mu + \partial_\nu *\hat{U}^{\mu\nu} \quad \text{in which} \quad *\hat{W}^\mu \equiv \hat{\Lambda}^\mu{}_\lambda \xi^\lambda = (\hat{\mathcal{L}}_\lambda^\mu + \hat{\mathcal{L}}_a^{\mu b} \omega_{\lambda b}^a) \xi^\lambda. \quad (3.11)$$

¹⁶Like in Julia and Silva [16]. I like this clever trick (JK)!

The vector $*\hat{J}^\mu$ is conserved, i.e. divergenceless, on-shell¹⁷ ($\hat{\mathcal{L}}_\lambda^\mu = \hat{\mathcal{L}}_a^{\mu b} = 0$). The value of the integral of $*\hat{J}^\mu$ on-shell over a hypersurface V is equal to the integral on the boundary S of V of $*\hat{U}^{\mu\nu}$. However the “generator” $*\hat{W}^\mu$ is not manifestly covariant; it depends on the choice of ω . In particular in the Palatini formalism $\theta^a_\mu = \delta^a_\mu$ and $\theta^\lambda_a \theta^b_\nu \omega^a_{\mu b} = \Gamma^\lambda_{\mu\nu}$. Thus $*\hat{J}^\mu$, which has much of the properties of a generator of conserved charges, is unfortunately not covariant. The way around this problem which we now describe is given in [19].

(ii) *Freedom of choice of the local frame*

In affine gravity theories the θ^a ’s are N arbitrary vectors which play the role of local tangent frames but the theories are invariant for arbitrary linear changes of basis $\theta^a(x) \rightarrow \Lambda^a_b(x)\theta^b(x)$. Therefore we can play the same game with this invariance as we played with the diffeomorphism invariance.

Let λ^a_b characterize an infinitesimal transformation denoted by δ_Λ . Thus,

$$\delta_\Lambda \theta^a_\mu = \lambda^a_b \theta^b_\mu \quad \text{implying that} \quad \delta_\Lambda \omega^a_{\mu b} = -\partial_\mu \lambda^a_b - \omega^a_{\mu c} \lambda^c_b + \omega^c_{\mu b} \lambda^a_c. \quad (3.12)$$

The Lagrangian density and the metric components are invariant under such transformations:

$$\delta_\Lambda \hat{\mathcal{L}} = 0 \quad , \quad \delta_\Lambda g^{\mu\nu} = 0. \quad (3.13)$$

Thus if δ is replaced by δ_Λ in (2.11), we obtain, after some rearrangement of factors,

$$(\hat{\mathcal{L}}_a^\mu \theta^b_\mu - \hat{\mathcal{L}}_c^{\mu b} \omega^c_{\mu a} + \hat{\mathcal{L}}_a^{\mu c} \omega^b_{\mu c}) \lambda^a_b - \hat{\mathcal{L}}_a^{\mu b} \partial_\mu \lambda^a_b = \partial_\nu (**\hat{J}_a^{\nu b} \lambda^a_b + **\hat{U}_a^{\nu\mu b} \partial_\mu \lambda^a_b), \quad (3.14)$$

The $\hat{\mathcal{D}}^\mu$ in (2.11) is linear in λ^a_b and its derivative and therefore the right-hand side in (3.14) is of the *form* written above.

We now play the same game as before by replacing $\lambda^a_b(x)$ with $\lambda^a_b(x)\epsilon(x)$ and expanding in terms of $\epsilon, \partial_\mu \epsilon$ and $D_{\mu\nu}\epsilon$, identifying the corresponding factors on both sides. The factors of ϵ reproduce of course (3.14). The factor of $D_{\mu\nu}\epsilon$ tells us that

$$**\hat{U}_a^{\nu\mu b} \lambda^a_b \equiv **\hat{U}^{\nu\mu} = -**\hat{U}^{\mu\nu}. \quad (3.15)$$

The equality of the factors of $\partial_\mu \epsilon$ leads to an equality similar to (3.11):

$$**\hat{J}_a^{\mu b} \lambda^a_b \equiv **\hat{J}^\mu = -\hat{\mathcal{L}}_a^{\mu b} \lambda^a_b + \partial_\nu (**\hat{U}^{\mu\nu}). \quad (3.16)$$

¹⁷With sources present the equations are of course different, except on the boundary.

Here is another non-covariant divergenceless vector on-shell $**\hat{J}^\mu$ with the properties of $*\hat{J}^\mu$, see (1.1). It is divergenceless but not covariant. If we now sum $*\mathbf{J}$ of (3.11) and $**\mathbf{J}$ just written, we obtain a new conserved current on shell with the same properties, which can be written as follows:

$$\hat{J}^\mu = *\hat{J}^\mu + **\hat{J}^\mu = \hat{\mathcal{L}}_\lambda^{\mu} \xi^\lambda + \hat{\mathcal{L}}_a^{\mu b} (\omega_{\nu b}^a \xi^\nu - \lambda_b^a) + \partial_\nu \hat{U}^{\mu\nu} \quad \text{with} \quad \hat{U}^{\mu\nu} = *\hat{U}^{\mu\nu} + **\hat{U}^{\mu\nu}. \quad (3.17)$$

(iii) *A covariant conserved current*

The final step in [18] entails the breaking of the local frame arbitrariness. The θ^a 's are kept fixed like they would in a Palatini formulation¹⁸, by taking

$$\delta_L \theta_\mu^a + \delta_\Lambda \theta_\mu^a = 0 \quad , \quad \partial_\mu \theta_\lambda^a = 0 \quad \text{and} \quad \theta_\mu^a = \delta_\mu^a \quad \Rightarrow \quad \lambda_b^a = -\theta_\lambda^a \theta_b^\mu \partial_\mu \xi^\lambda \quad , \quad \theta_a^\lambda \theta_\nu^b \omega_{\mu b}^a = \Gamma_{\mu\nu}^\lambda. \quad (3.18)$$

Substituting the last two equalities in the second term of (3.17) on the right-hand side gives:

$$\hat{\mathcal{L}}_a^{\mu b} \omega_{\lambda b}^a \xi^\lambda + \hat{\mathcal{L}}_\lambda^{\mu\nu} \partial_\nu \xi^\lambda = \hat{\mathcal{L}}_\eta^{\mu\nu} \Gamma_{\lambda\nu}^\eta \xi^\lambda + \hat{\mathcal{L}}_\nu^{\mu\lambda} \partial_\lambda \xi^\nu = \hat{\mathcal{L}}_\lambda^{\mu\nu} D_\nu \xi^\lambda. \quad (3.19)$$

So, $\hat{\mathbf{J}}$ is now of the form

$$\hat{J}^\mu = \hat{W}^\mu + \partial_\nu \hat{U}^{\mu\nu} \quad \text{with} \quad \hat{W}^\mu = \hat{\mathcal{L}}_\lambda^{\mu} \xi^\lambda + \hat{\mathcal{L}}_\lambda^{\mu\nu} D_\nu \xi^\lambda. \quad (3.20)$$

The covariant \hat{J}^μ is conserved on-shell ($\hat{W}^\mu = 0$) and the charge

$$Q = \oint_S \hat{U}^{\mu\nu} dS_{\mu\nu}. \quad (3.21)$$

Q is, of course, not yet well defined because the superpotential is not well-defined. We come back to this problem with an answer below. But we shall first show that the same current is obtained by applying Noether's prescription to the same Lagrangian density in the Palatini formalism with \mathbf{g} and $\mathbf{\Gamma}$ as independent fields without introducing an additional set of N vectors θ . We shall follow [22] in this matter.

¹⁸Or, if one prefers, in an Einstein-Cartan formalism ($\theta_\mu^a = e_\mu^a$).

4 Symmetries and conserved currents in Palatini formalism

Conservation laws related to diffeomorphism invariance are derived in the same way as in affine gravity. The difference lies in an extra derivative of ξ . The Lie derivative $\delta_L \hat{\mathcal{L}} = \partial_\mu (\hat{\mathcal{L}} \xi^\mu)$ as before. The Lie derivative of $g^{\mu\nu}$ is the same as in (3.1). However the Lie derivative of the Γ 's contains second order derivatives and is a tensor:

$$\delta_L \Gamma^\lambda_{\mu\nu} = -R^\lambda_{(\mu\nu)\eta} \xi^\eta + D_{(\mu\nu)} \xi^\lambda. \quad (4.1)$$

The result is that instead of an identity like (3.4) there is one additional term on both sides in $D_{\mu\nu} \xi^\lambda$. Thus after replacing δ by δ_L in (2.4), using (3.1) and (4.1), we obtain an identity of the following form:

$$\hat{\Lambda}_\lambda \xi^\lambda + \hat{\Lambda}_\lambda^\mu D_\mu \xi^\lambda + \hat{\mathcal{L}}_\lambda^{\mu\nu} D_{\mu\nu} \xi^\lambda = \partial_\nu \left(\hat{J}_\lambda^\nu \xi^\lambda + \bullet \hat{U}_\lambda^{\nu\mu} D_\mu \xi^\lambda + \hat{V}_\lambda^{\nu\rho\sigma} D_{(\rho\sigma)} \xi^\lambda \right) \quad (4.2)$$

in which

$$\hat{\Lambda}_\lambda = * \hat{\mathcal{L}}_{\mu\nu} D_\lambda g^{\mu\nu} - \hat{\mathcal{L}}_\eta^{\mu\nu} R^\eta_{\mu\nu\lambda} \quad , \quad \hat{\Lambda}_\lambda^\mu = -2 * \hat{\mathcal{L}}_\lambda^\mu. \quad (4.3)$$

Now in (4.2) we replace $\xi(x)$ by $\xi(x)\epsilon(x)$ and, as before, we expand in terms of $\epsilon, \partial_\mu \epsilon, D_{\mu\nu} \epsilon$ and $D_{\lambda\mu\nu} \epsilon$. On the left-hand side we have

$$(\hat{\Lambda}_\lambda \xi^\lambda + \hat{\Lambda}_\lambda^\mu D_\mu \xi^\lambda + \hat{\mathcal{L}}_\lambda^{\mu\nu} D_{\mu\nu} \xi^\lambda) \epsilon + (\hat{\Lambda}_\lambda^\mu \xi^\lambda + 2 \hat{\mathcal{L}}_\lambda^{\mu\nu} D_\nu \xi^\lambda) D_\mu \epsilon + (\hat{\mathcal{L}}_\lambda^{\mu\nu} \xi^\lambda) D_{\mu\nu} \epsilon. \quad (4.4)$$

On the right hand side we have however this:

$$\begin{aligned} \partial_\nu (\hat{J}^\nu \epsilon + \bullet \hat{U}^{\nu\mu} \partial_\mu \epsilon + \hat{V}^{\nu\lambda\mu} D_{\lambda\mu} \epsilon) = \\ (\partial_\nu \hat{J}^\nu) \epsilon + (\hat{J}^\mu + \partial_\nu \bullet \hat{U}^{\nu\mu}) \partial_\mu \epsilon + (\bullet \hat{U}^{\nu\mu} + D_\lambda \hat{V}^{\lambda\mu\nu}) D_{\mu\nu} \epsilon + \hat{V}^{\lambda\mu\nu} D_{\lambda\mu\nu} \epsilon. \end{aligned} \quad (4.5)$$

The factors of ϵ and its derivatives on the left-hand side (4.4) are identical to the factors of those derivatives on the right-hand side (4.5). The identity “left-hand side (4.4) \equiv right-hand side (4.5)” holds in arbitrary coordinates. We may take coordinates in which at any particular point the metric is Minkowski and the Γ 's are equal to zero. At that point the last term in (4.5) becomes $\hat{V}^{\lambda\mu\nu} \partial_{\lambda\mu\nu} \epsilon$. It must be zero because there is no such term on the left-hand side (4.4). This implies that $\hat{V}^{\lambda\mu\nu}$, totally symmetrized in all its indices, equals zero or that

$$\hat{V}^{(\lambda\mu\nu)} = \frac{1}{3} (\hat{V}^{\lambda\mu\nu} + \hat{V}^{\mu\nu\lambda} + \hat{V}^{\nu\lambda\mu}) = 0 \quad \text{and, see (4.2),} \quad \hat{V}^{\lambda\mu\nu} = \hat{V}^{\lambda\nu\mu}. \quad (4.6)$$

Any \mathbf{V} that has these properties has also the following property which is easily proven:

$$\hat{V}^{\lambda\mu\nu} D_{\lambda\mu\nu}\epsilon = -\frac{2}{3}\hat{V}^{[\mu\nu]\lambda}R^\eta_{\lambda\mu\nu}\partial_\eta\epsilon. \quad (4.7)$$

With this equality (4.7), we may rewrite the right-hand side in (4.5) as follows

$$(\partial_\nu \hat{J}^\nu)\epsilon + (\hat{J}^\mu + \partial_\nu \bullet \hat{U}^{\nu\mu} - \frac{2}{3}\hat{V}^{[\rho\sigma]\nu}R^\mu_{\nu\rho\sigma})\partial_\mu\epsilon + (\hat{\mathcal{L}}_\lambda^{\mu\nu})D_{\mu\nu}\epsilon. \quad (4.8)$$

The left-hand side (4.4) is thus also identical to the right-hand side in the form (4.8) for any values of ϵ : the factors of ϵ and its derivatives in (4.4) are identical to the corresponding factors in (4.8).

The factors of ϵ are obtained by setting $\epsilon = 1$ which reproduces thus exactly (4.2) from which we learn nothing new. The two remaining identities obtained by identifying the factors of $\partial_\mu\epsilon$ and $D_{\mu\nu}\epsilon$, are respectively

$$-2*\hat{\mathcal{L}}_\lambda^\mu\xi^\lambda + 2\hat{\mathcal{L}}_\lambda^{\mu\nu}D_\nu\xi^\lambda = \hat{J}^\mu + D_\nu\bullet\hat{U}^{\nu\mu} - \frac{2}{3}\hat{V}^{[\rho\sigma]\nu}R^\mu_{\nu\rho\sigma}, \quad (4.9)$$

$$\hat{\mathcal{L}}_\lambda^{\mu\nu}\xi^\lambda = \bullet\hat{U}^{(\mu\nu)} + D_\lambda\hat{V}^{\lambda\mu\nu}. \quad (4.10)$$

We shall decompose $\bullet\hat{U}^{\nu\mu}$ appearing in (4.9) into a symmetric $\bullet\hat{U}^{(\nu\mu)}$ and an anti-symmetric part $\bullet\hat{U}^{[\nu\mu]}$ in $\mu\nu$. The symmetric part will be replaced by its value given in (4.10), i.e.

$$\bullet\hat{U}^{(\mu\nu)} = \hat{\mathcal{L}}_\lambda^{\mu\nu}\xi^\lambda - D_\lambda\hat{V}^{\lambda\mu\nu}. \quad (4.11)$$

The right-hand side of (4.9) then becomes

$$\hat{J}^\mu + \partial_\nu \bullet\hat{U}^{[\nu\mu]} + \hat{\mathcal{L}}_\lambda^{\mu\nu}\xi^\lambda - D_\lambda\hat{V}^{\lambda\mu\nu} - \frac{2}{3}\hat{V}^{[\rho\sigma]\nu}R^\mu_{\nu\rho\sigma}. \quad (4.12)$$

However, the last two terms combine to give

$$-D_\lambda\hat{V}^{\lambda\mu\nu} - \frac{2}{3}\hat{V}^{[\rho\sigma]\nu}R^\mu_{\nu\rho\sigma} = \frac{2}{3}\partial_\nu \left(D_\lambda\hat{V}^{[\mu\nu]\lambda} \right), \quad (4.13)$$

so that the right-hand side of (4.9), i.e. (4.12), is now of the form

$$\hat{J}^\mu - \partial_\nu \hat{U}^{\mu\nu} + \hat{\mathcal{L}}_\lambda^{\mu\nu}\xi^\lambda \quad \text{in which} \quad \hat{U}^{\mu\nu} = -\hat{U}^{\nu\mu} = \bullet\hat{U}^{[\mu\nu]} - \frac{2}{3}D_\lambda\hat{V}^{[\mu\nu]\lambda}. \quad (4.14)$$

Since the left-hand side of (4.9) is equal to (4.12), we have now

$$\hat{J}^\mu = \hat{W}_P^\mu + \partial_\nu \hat{U}^{[\mu\nu]} \quad \text{in which} \quad \hat{W}_P^\mu = (-2*\hat{\mathcal{L}}_\nu^\mu - D_\lambda\hat{\mathcal{L}}_\nu^{\mu\lambda})\xi^\nu + \hat{\mathcal{L}}_\lambda^{\mu\nu}D_\nu\xi^\lambda \quad (4.15)$$

This expression is obviously the same as (3.20) in affine gravity since, if we take into account the identity (2.18),

$$\hat{W}_P^\mu = \hat{W}^\mu = \hat{\mathcal{L}}_\lambda^\mu \xi^\lambda + \hat{\mathcal{L}}_\lambda^{\mu\nu} D_\nu \xi^\lambda, \quad (4.16)$$

which is indeed (3.20).

The passage from affine gravity to the Palatini formalism is implicit in the work of Julia and Silva [18]. One difference is that the whole layout is here explicit and in tensorial notations.

5 Superpotentials and charges in Lovelock gravity theories

(i) Hamiltonian covariant formalism

With an expression for the current conserved on-shell (4.15) we may write the covariant Hamiltonian H . We keep the letter H for in the end we are mostly interested in the total mass energy. However, at this stage we shall not specialize ξ to an asymptotic timelike Killing vector. Thus, in the Palatini formalism,

$$H = \int_V \hat{W}^\mu dV_\mu + \oint_S \hat{U}^{\mu\nu} dS_{\mu\nu} = \int_V \left(\hat{\mathcal{L}}_\lambda^\mu \xi^\lambda + \hat{\mathcal{L}}_\lambda^{\mu\nu} D_\nu \xi^\lambda \right) dV_\mu + \oint_S \hat{U}^{\mu\nu} dS_{\mu\nu}. \quad (5.1)$$

We can equally use the field components of affine gravity, before breaking the gauge invariance. The calculation is easily seen to lead to the same result once gauge invariance is broken. From (3.6) we know that $\hat{\mathcal{L}}_\lambda^\mu$ is an ordinary derivative which depends on \mathbf{g} and \mathbf{B} only. On the other hand, in Lovelock gravity, $\hat{\mathcal{L}}_\lambda^{\mu\nu}$ are variational derivatives, see (2.5), which contain not only \mathbf{g} and \mathbf{B} but are linear and homogeneous in derivatives of the metric components $D_\lambda g^{\mu\nu}$ but contain no derivatives of \mathbf{B} . The covariant Hamiltonian equations are obtained from the variation of H which reduces thus to:

$$\delta H = \int_V \left[\frac{\delta \hat{W}^\mu}{\delta g^{\rho\sigma}} \delta g^{\rho\sigma} + \frac{\delta \hat{W}^\mu}{\delta \Gamma_{\rho\sigma}^\lambda} \delta \Gamma_{\rho\sigma}^\lambda \right] dV_\mu + \oint_S \left[\frac{\partial \hat{W}^\mu}{(\partial_\nu g^{\rho\sigma})} \delta g^{\rho\sigma} + \frac{\partial \hat{\mathcal{L}}_\eta^{\mu\xi\eta}}{\partial (\partial_\nu \Gamma_{\rho\sigma}^\lambda)} \delta \Gamma_{\rho\sigma}^\lambda + \delta \hat{U}^{\mu\nu} \right] dS_{\mu\nu} \quad (5.2)$$

It is not at all obvious that the term in front of $\delta \hat{U}^{\mu\nu}$ in the brackets on the right hand side is antisymmetrical in $\mu\nu$, however it is. The prove goes as follows: consider the left hand side of (3.4) which is a divergence. Its variational derivative with respect to any field component, say y^A , is thus equal to zero. This holds for any ξ . Replace ξ by $\epsilon(x)\xi$. It holds also for any

$\epsilon(x)$. The factor of $D_{\mu\nu}\epsilon$ must thus be equal to zero. This factor is $\partial\hat{W}^{(\mu}/\partial[\partial_{\nu)}y^A]$. Thus the factors in the bracket on the right hand side are anti-symmetric in $\mu\nu$.

(ii) *Dirichlet boundary conditions and the superpotential*

Following Julia and Silva the functional equation for the superpotential is given by equating the second bracket to zero:

$$\frac{\partial\hat{W}^\mu}{(\partial_\nu g^{\rho\sigma})}\delta g^{\rho\sigma} + \frac{\partial\hat{\mathcal{L}}_\eta^{\mu\xi\eta}}{\partial(\partial_\nu\Gamma_{\rho\sigma}^\lambda)}\delta\Gamma_{\rho\sigma}^\lambda + \delta\hat{U}^{\mu\nu} = 0. \quad (5.3)$$

Superpotentials must satisfy all boundary conditions. The most common conditions in Einstein's theory of gravity are the Dirichlet boundary conditions $g^{\rho\sigma}|_S = \bar{g}^{\rho\sigma}$, or equivalently $\delta g^{\rho\sigma}|_S = 0$. We shall also identify the Γ 's on the boundary with Christoffel symbols which is equivalent to taking $D_\lambda g^{\rho\sigma}|_S = 0$ or $\delta(D_\lambda g^{\rho\sigma})|_S = 0$. As a consequence on the boundary $B^\lambda_{\sigma\mu\nu}|_S = \bar{R}^\lambda_{\sigma\mu\nu}$, the Riemann tensor of the background, and $\delta B^\lambda_{\sigma\mu\nu}|_S = 0$. Finally the field equations themselves must remain satisfied. Thus, for isolated sources the boundary conditions are:

$$\delta g^{\rho\sigma}|_S = \delta(D_\lambda g^{\rho\sigma})|_S = \delta B^\lambda_{\sigma\mu\nu}|_S = \delta\hat{\mathcal{L}}_\lambda^\mu = \delta\hat{\mathcal{L}}_\lambda^{\mu\nu} = 0. \quad (5.4)$$

Equations (5.3) reduce thus to

$$\frac{\partial(\hat{\mathcal{L}}_\eta^{\mu\xi\eta})}{\partial(\partial_\nu\Gamma_{\rho\sigma}^\lambda)}\delta\Gamma_{\rho\sigma}^\lambda + \delta\hat{U}^{\mu\nu} = 0. \quad (5.5)$$

The factors of $\delta\Gamma_{\rho\sigma}^\lambda$ in (5.5) are entirely defined on the boundary. As far as the differential equation for \mathbf{U} is concerned, we may thus integrate equation (5.5), and obtain an expression linear in $\Gamma_{\rho\sigma}^\lambda$. The “constant of integration” will be *minus* the same expression with $\Gamma_{\rho\sigma}^\lambda$ replaced by $\bar{\Gamma}_{\rho\sigma}^\lambda$ for reasons explained in the Introduction. The formula for the superpotential is thus

$$\hat{U}^{\mu\nu} = -\frac{\partial(\hat{\mathcal{L}}_\eta^{\mu\xi\eta})}{\partial(\partial_\nu\Gamma_{\rho\sigma}^\lambda)}\Delta^\lambda_{\rho\sigma} \quad \text{with} \quad \Delta^\lambda_{\rho\sigma} = \Gamma_{\rho\sigma}^\lambda - \bar{\Gamma}_{\rho\sigma}^\lambda. \quad (5.6)$$

Finally, since $\hat{\mathcal{L}}$ is a function of \mathbf{g} and \mathbf{B} , we may also rewrite (5.5) like this:

$$\hat{U}^{\mu\nu} = 2\frac{\partial(\hat{\mathcal{L}}_\eta^{\mu\xi\eta})}{\partial B^\lambda_{\rho\sigma\nu}}\Delta^\lambda_{\rho\sigma}. \quad (5.7)$$

This formula for the covariant superpotential is a blend of two derivatives: one with respect to $\boldsymbol{\theta}$ at $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ constant in affine gravity giving the generator $\hat{\mathcal{L}}_\eta^{\mu\xi\eta}$ and one with respect

to $B^\lambda_{\rho\sigma\nu}$ at \mathbf{g} constant in the Palatini representation. As far as we know this result is new, simple and compact. The corresponding charge¹⁹ Q is

$$Q = \oint_S \hat{U}^{\mu\nu} dS_{\mu\nu} \quad (5.8)$$

in which $\hat{U}^{\mu\nu}$ is defined in (5.7).

6 Some examples

(i) Charges in N dimensional Einstein's theory of gravitation

This simple example will show how the machinery works. The Lagrangian is

$$\hat{\mathcal{L}}_1 = \frac{1}{2\kappa} \hat{B} = \frac{1}{2\kappa} \hat{g}^{\mu\nu} B_{\mu\nu} = \frac{1}{4\kappa} (\delta^\rho_\lambda \hat{g}^{\sigma\nu} - \delta^\sigma_\lambda \hat{g}^{\rho\nu}) B^\lambda_{\nu\rho\sigma} = \frac{1}{2\kappa} \delta^\rho_\lambda \hat{g}^{\sigma\nu} B^\lambda_{\nu[\rho\sigma]}. \quad (6.1)$$

In four dimensions we take as usual for κ Einstein's coupling constant $\kappa = \frac{8\pi G}{c^4}$. In N dimensions

$$\kappa = \frac{2S_{N-2}G_N}{c^4}, \quad (6.2)$$

where S_{N-2} is the surface of a unit sphere of dimension $(N-2)$ and G_N is a gravity coupling constant. In affine gravity, the field components are γ^{ab} , θ^a_μ and $\omega^a_{\mu b}$ in terms of which, see (2.6) and (2.9):

$$\hat{\mathcal{L}}_1 = \frac{1}{2\kappa} |\theta| \theta^a_\mu \theta^b_\nu \left(\sqrt{-\gamma} B^{[ab]}_{[\mu\nu]} \right) \text{ where } \theta = \det(\theta^a_\mu), \gamma = \det(\gamma_{ab}) \text{ and } B^{ab}_{\mu\nu} = \gamma^{bc} B^a_{c\mu\nu}. \quad (6.3)$$

The term in parenthesis is independent of the θ 's. The calculation of the derivative with respect to θ^a_μ is thus very simple. One has, however, to be careful about symmetrizations and anti-symmetrizations of indices. The quantity that interests us, the generator appearing in (5.6), is here ${}_1\hat{\mathcal{L}}^\mu_\eta \xi^\eta$ which in a condensed form may be written as

$${}_1\hat{\mathcal{L}}^\mu_\eta \xi^\eta = \frac{1}{2\kappa} \left(\xi^\mu \delta^{[\sigma}_\lambda \hat{g}^{\nu]\rho} + \xi^\sigma \delta^{[\nu}_\lambda \hat{g}^{\mu]\rho} + \xi^\nu \delta^{[\mu}_\lambda \hat{g}^{\sigma]\rho} \right) B^\lambda_{\rho\sigma\nu}. \quad (6.4)$$

From this follows that the superpotential \mathbf{U}_1 is

$$\hat{U}_1^{\mu\nu} = \frac{1}{2\kappa} \left(\xi^{[\mu} \hat{g}^{\nu]\rho} \Delta^\sigma_{\rho\sigma} + \xi^{[\nu} \hat{g}^{\sigma\rho} \Delta^\mu_{\rho\sigma} + \xi^\sigma \hat{g}^{\rho[\mu} \Delta^{\nu]}_{\rho\sigma} \right). \quad (6.5)$$

¹⁹The superpotential given in [22] is not the same as the one given in (5.7). The reason is that we used an equality (3.20) (in [22]) which does not correspond to a covariant generalization of the Regge-Teitelboim procedure. It is rather equivalent to what amounts in Julia and Silva [18] to take $X \neq 0$ in their equation (10). The superpotential we found in [22] gives however correctly the charges for backgrounds that are maximally symmetrical which is the most common case and which explain that all the applications described in [22] are correct. The claim of the paper that it generalizes the Regge Teitelboim method along the line described by Julia and Silva is, however, wrong.

Notice the circular symmetry of indices μ, σ, ν in (6.4) and μ, ν, σ in (6.5). In exactly this form, but in Minkowski coordinates on a flat background and for $\xi^\mu = \{1, 0, 0, 0\}$, the superpotential for the total mass energy in Einstein's theory in 4 dimensions was published by von Freud [35] in 1939.

A more standard form is obtained by noticing that

$$\xi^\sigma \hat{g}^{\rho[\mu} \Delta^{\nu]}_{\rho\sigma} = D^{[\mu} \hat{\xi}^{\nu]} - \overline{D^{[\mu} \hat{\xi}^{\nu]}} \quad (6.6)$$

and then writing $\hat{U}_1^{\mu\nu}$ like this:

$$\hat{U}_1^{\mu\nu} = \frac{1}{2\kappa} \left(D^{[\mu} \hat{\xi}^{\nu]} - \overline{D^{[\mu} \hat{\xi}^{\nu]}} \right) + \frac{1}{2\kappa} \xi^{[\mu} \hat{k}^{\nu]} \quad \text{with} \quad \hat{k}^\nu = \hat{g}^{\nu\rho} \Delta^\sigma_{\rho\sigma} - \hat{g}^{\rho\sigma} \Delta^\nu_{\rho\sigma}. \quad (6.7)$$

In this form we recognize the KBL [23] superpotential²⁰. The first term $\frac{1}{2\kappa} D^{[\mu} \hat{\xi}^{\nu]}$ is half the Komar [25] superpotential. For applications formula (6.5) is simpler. Various physical properties including standard results for charges derived from the KBL superpotential have been reviewed in [18]. It gives among other things the Bondi mass [21] for radiating systems in four dimensions. The KBL superpotential is thus valid at both spatial and null infinity and on any background that satisfies the boundary conditions. Here is a new application of the superpotential at null infinity in five dimensions. We have calculated the mass loss using the asymptotic structure of the solution of Einstein's equations given by Tanabe, Tanahashi and Shiromizu [33], see also [34], in ‘‘Bondi coordinates’’ and obtained their formula for the mass loss²¹. Some details of the calculations are given in Appendix A.

(ii) Superpotential of Gauss-Bonnet Gravity in $N > 4$ dimensions

Einstein-Gauss-Bonnet theories of gravity are usually considered in combination with Einstein's Lagrangian [see sub-section (iii) below]. Since we obtained already $\hat{\mathbf{U}}_1$ we shall consider $\hat{\mathbf{U}}_2$ derived from the Lagrangian density $\hat{\mathcal{L}}_2$ which we shall write in a form similar to (6.3) the most convenient to calculate the generator ${}_2\hat{\mathcal{L}}_\eta^\mu \xi^\eta$ and which we took from Lovelock's paper of 1971 [26]:

$$\hat{\mathcal{L}}_2 = \frac{1}{2^2\kappa} |\theta| \theta_a^\mu \theta_b^\nu \theta_c^\rho \theta_d^\sigma \left(\sqrt{-\gamma} B^{[ab]}_{[\mu\nu]} B^{[cd]}_{[\rho\sigma]} \right). \quad (6.8)$$

The terms within parentheses depends on γ 's and ω 's and not on θ . The θ 's are all in front of the parenthesis. Equation (6.8) provides the generator ${}_2\hat{\mathcal{L}}_\eta^\mu \xi^\eta$ through ordinary partial

²⁰On a flat background this expression was given in [20].

²¹Adjusted to our coupling constant in front of the Lagrangian and our units.

derivatives of (6.8) with respect to the θ 's. The result is worth rewriting in terms of \mathbf{g} 's and \mathbf{B} 's, since the superpotential $\hat{\mathbf{U}}_2$ is also obtained by ordinary partial derivatives of the generator ${}_2\hat{\mathcal{L}}_\eta^\mu \xi^\eta$ with respect \mathbf{B} 's at fixed \mathbf{g} 's:

$${}_2\hat{\mathcal{L}}_\eta^\mu \xi^\eta = \hat{\Theta}_{\lambda\zeta}^{\mu\eta\tau\mu'\nu'\rho'\sigma'} B_{\eta\mu'\nu'}^\lambda B_{\tau\rho'\sigma'}^\zeta \quad (6.9)$$

in which

$$\hat{\Theta}_{\lambda\zeta}^{\mu\eta\tau\mu'\nu'\rho'\sigma'} = \frac{\sqrt{-g}}{2^2\kappa} \left[\left(\xi^\mu \delta_\lambda^{[\mu'} g^{\nu']\eta} + \xi^{\mu'} \delta_\lambda^{[\nu'} g^{\mu]\eta} + \xi^\nu \delta^{[\mu} g^{\mu']\eta} \right) \delta_\zeta^{[\rho'} g^{\sigma']\tau} + \delta_\lambda^{[\mu'} g^{\nu']\eta} \left(\xi^{\rho'} \delta_\zeta^{[\sigma'} g^{\mu]\tau} - \xi^{\sigma'} \delta_\zeta^{[\rho'} g^{\mu]\tau} \right) \right]. \quad (6.10)$$

The resulting $\hat{\mathbf{U}}_2$, see (5.7), contains now 9 triplets with circular symmetry, altogether 27 terms linear in the curvature tensor:

$$\begin{aligned} \kappa \hat{U}_2^{\mu\nu} = & 3 \left[\hat{\xi}^{(\rho} R^{\mu\nu)\sigma}{}_\lambda \Delta^\lambda_{\rho\sigma} + \left(g^{\rho(\mu} G_\eta^\nu \Delta^\sigma_{\rho\sigma} - g^{\rho(\nu} G_\eta^\mu \Delta^\sigma_{\rho\sigma} \right) \hat{\xi}^\eta \right] \\ & + 3 \left(g^{\rho(\sigma} R^{\mu\nu)}{}_{\lambda\eta} \hat{\xi}^\eta + R_\lambda^{(\mu} \hat{\xi}^{\rho} g^{\nu)\sigma} - R_\lambda^{(\nu} \hat{\xi}^{\rho} g^{\mu)\sigma} \right) \Delta^\lambda_{\rho\sigma} \\ & - 3 \left(\Delta^{(\rho}{}_{\rho\sigma} R^{\mu\nu)\sigma}{}_\eta \hat{\xi}^\eta + R^{\rho(\mu} \Delta^\nu{}_{\rho\sigma} \hat{\xi}^\sigma - R^{\rho(\nu} \Delta^\mu{}_{\rho\sigma} \hat{\xi}^\sigma \right). \end{aligned} \quad (6.11)$$

The parenthesis represent circular permutations as in (4.6), which explains all those 3's. If we separate \hat{G}_η^σ into Ricci and scalar curvature parts there are 11 instead of 9 triplets and altogether 33 terms. An application of this complicated formula is given in the next subsection.

(iii) *Einstein-Gauss-Bonnet theories of gravity with maximally symmetric boundaries*

In these theories of gravitation, the Lagrangian density is a linear combination of the Einstein-Hilbert Lagrangian $\hat{\mathcal{L}}_1$ (a Lovelock Lagrangian of order 1) and the Gauss-Bonnet Lagrangian $\hat{\mathcal{L}}_2$ (a Lovelock Lagrangian of order 2)

$$\hat{\mathcal{L}}_{2'} = \alpha_1 \hat{\mathcal{L}}_1 + \alpha_2 \hat{\mathcal{L}}_2, \quad (6.12)$$

in which α_1 and α_2 are self-coupling constants to be found experimentally. The corresponding superpotential is thus

$$\hat{U}_{2'}^{\mu\nu} = \alpha_1 \hat{U}_1^{\mu\nu} + \alpha_2 \hat{U}_2^{\mu\nu}. \quad (6.13)$$

In past applications backgrounds have mostly been flat or anti-de Sitter, i.e. maximally symmetric spacetimes. In such spacetimes:

$$R^\lambda_{\nu\rho\sigma} = \frac{2R}{N(N-1)} \delta^\lambda_{[\rho} \bar{g}_{\sigma]\nu} \quad \text{and} \quad R_{\mu\nu} = \frac{R}{N} \bar{g}_{\mu\nu}. \quad (6.14)$$

This makes $\hat{\mathbf{U}}_2$ considerably simpler:

$$\hat{U}_2^{\mu\nu} = \frac{(N-3)(N-4)}{N(N-1)} R \hat{U}_1^{\mu\nu}. \quad (6.15)$$

We have applied this result to calculate the mass of the Einstein-Gauss-Bonnet black holes found by Deser and Tekin [7] with a background curvature

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2} (\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}) \quad \text{where} \quad l^2 = -\frac{(N-1)(N-2)}{2\Lambda}, \quad \Lambda < 0. \quad (6.16)$$

in the case considered in [7], $\alpha_1 = 1$ and

$$\alpha_2 = \frac{-2l^2}{(N-3)(N-4)}, \quad (6.17)$$

one obtains the surprising result that

$$\hat{U}_{2'}^{\mu\nu} = -\hat{U}_1^{\mu\nu}! \quad (6.18)$$

But this is indeed the correct result and it gives the positive mass found in Deser and Tekin by a quite different but reliable method.

(iv) Lovelock Gravity in any number of dimensions

The Lovelock Lagrangian density of order n in N dimensions is as follows:

$$\hat{\mathcal{L}}_n = \frac{\sqrt{-g}}{2^n \kappa} \sum_{n=0}^{n=p-1} \delta_{\mu_1\mu_2\cdots\mu_{2p}}^{\nu_1\nu_2\cdots\nu_{2p}} B^{\mu_1\mu_2}_{\nu_1\nu_2} \cdots B^{\mu_{2p-1}\mu_{2p}}_{\nu_{2p-1}\nu_{2p}} \quad (6.19)$$

with

$$1 \leq n \leq p-1 \text{ where } p = \frac{1}{2}N \text{ (for } N \text{ even)}, \quad p = \frac{1}{2}(N+1) \text{ (for } N \text{ odd)} \quad \text{and} \quad \delta_{\mu_1\mu_2\cdots\mu_{2p}}^{\nu_1\nu_2\cdots\nu_{2p}} = \det(\boldsymbol{\delta}). \quad (6.20)$$

In 1985 Zwiebach [37] speculated that the Lagrangian that would ultimately describe gravitational interactions and more fundamentally string interactions should have an infinite number of terms. Each $\hat{\mathbf{U}}_n$ would give a finite contribution. In particular for spacetimes that are asymptotically maximally symmetric,

$$\hat{U}_n^{\mu\nu} = n \frac{(N-2n)[N-(2n-1)] \cdots (N-4)(N-3)}{[N(N-1)]^{n-1}} R^{n-1} \hat{U}_1^{\mu\nu}. \quad (6.21)$$

and for $N \rightarrow \infty$, the factor containing N is equal to one. Thus, if the Lagrangian density

$$\hat{\mathcal{L}}_\infty = \sum_{n=0}^{\infty} \alpha_n \hat{\mathcal{L}}_n, \quad (6.22)$$

the superpotential is

$$\hat{U}_\infty^{\mu\nu} = \left(\sum_{n=0}^{\infty} n \alpha_n R^n \right) \hat{U}_1^{\mu\nu}. \quad (6.23)$$

This sum may converge for suitable α 's.

7 Some comments

a) Regarding $\hat{U}_1^{\rho\sigma}$ as given by equation (6.5). On a flat background, in Minkowski coordinates and with ξ the Killing vectors of spacetime translations, $\hat{U}_1^{\rho\sigma}$, as we observed, is exactly the superpotential that Freud [35] found, more than 70 years ago, to calculate mass-energy and total linear momentum in Einstein's theory. One wonders why Freud did not calculate the angular momentum on the same occasion.

b) We emphasized several times that the superpotential holds at null as well as at spatial infinity and that $\hat{U}_1^{\rho\sigma}$ at null infinity gives the Bondi mass loss in four dimensions [21] as well as in 5 dimensions as we show in Appendix A. We are not aware of a published work on radiating fields in Einstein-Gauss-Bonnet theories. $\hat{U}_{2'}^{\rho\sigma}$ is, of course, the superpotential appropriate for calculating the Bondi mass loss in that case.

c) We imposed Dirichlet boundary conditions and found the KBL superpotential in N dimensional Einstein's gravity. Julia and Silva showed that by imposing Neumann boundary conditions we are led to Komar's superpotential²². Neumann boundary conditions have been considered in recent works. See for instance a paper by Kofinas and Olea [24] on Lovelock anti-de Sitter gravity. We presume that imposing Neumann boundary conditions would have led to Noether charges as considered in Iyer and Wald [13].

d) Jacobson and Myers [15] used the first law of thermodynamics to define mass-energy of Lovelock black holes. Mass-energy has, however, little to do with thermodynamics because asymptotic spacetimes ignore the source of gravity. A direct calculation of the mass-energy of Lovelock black-holes, independent of thermodynamic considerations, is naturally provided by the superpotential $\hat{U}^{\rho\sigma}$. An approach, similar to that of Jacobson and Myers, was used by Gibbons *et al* [9] to find the mass of Kerr-anti-de Sitter black holes. Needless to say the same mass-energy is obtained with the KBL superpotential $\hat{U}_1^{\rho\sigma}$, see [6].

e) We dealt with Lovelock Lagrangians whose variational derivatives have no derivatives of the curvature tensor. Conservation laws have, however, been considered for non-Lovelock

²²A generalization of Komar's superpotential, that does not need a background, to any type of boundary conditions, can be found in several papers of Obukhov and Rubilar's ; see for instance [28]. The role of backgrounds is here replaced by another ingredient, "generalized" Lie derivatives.

Lagrangians by Deser and Tenkin [7] at spatial infinity and by Wald and Zoupas [36] at null infinity. A covariant generalization of Regge and Teitelboim principle to a Palatini or affine gravity formulation needs yet to be worked out but do not appear to be trivial unless one introduces auxiliary fields in order to reduce second order equations to first order one which is always possible while preserving the gauge symmetries, see [17].

Acknowledgments

G. I. L. is very grateful to Tanabe for his advice in calculating the Bondi mass.

Appendix

A The Bondi loss of mass in 5 dimensional Einstein gravity

Tanabe, Tanahishi and Shiromizu [33] calculated the asymptotic solution of Einstein's equations in five dimensions at null infinity. They used coordinates similar to those of Bondi and calculated, among other things, the Bondi loss of mass-energy. We have calculated the mass loss from our formula (6.4) for $\hat{\mathbf{U}}_1$ and have found, of course, the same result. Here we use their notations up to minor changes with their signature of the metric and give only the quantities needed for the calculation. The Bondi coordinates are $x^\mu = \{u, r, x^A\}$ thus with $x^A = \{\theta, \varphi, \psi\}$, $A, B, \dots = 2, 3, 4$ which are the coordinates of a unit sphere of dimension 3, S_3 . In these coordinates, the line element may be written

$$ds^2 = \left(-\frac{Ve^B}{r^2} + r^2 h_{AB} U^A U^B \right) du^2 - 2e^B dudr + r^2 h_{AB} (dx^A + U^A du)(dx^B + U^B du). \quad (\text{A.1})$$

V, B, U^A depend on all coordinates. The asymptotic expansion in powers of r is as follows:

$$V = r^2 + V_1 r^{1/2} - \mathcal{E} + \mathcal{O}(r^{-1/2}) \quad , \quad B = \frac{B_1}{r^3} + \mathcal{O}(r^{-4}) \quad , \quad U^A = \frac{U_1^A}{r^{5/2}} + \mathcal{O}(r^{-3}), \quad (\text{A.2})$$

V_1, B_1, U_1^A and h_{AB} are functions of $\{u, x^A\}$ only:

$$\begin{aligned}
V_1 &= -\frac{2}{3} \left[\frac{1}{\sin 2\theta} \partial_\theta (\sin 2\theta U_1^\theta) + \partial_\varphi U_1^\varphi + \partial_\psi U_1^\psi \right], \\
B_1 &= -\frac{1}{8} (C_{11}^2 + C_{11}C_{21} + C_{12}^2 + D_{11}^2 + D_{12}^2 + D_{13}^2), \\
U_1^\theta &= \frac{2}{5} \left[\frac{(C_{11} - C_{12})}{\tan \theta} - (2C_{11} + C_{12}) \tan \theta + \partial_\theta C_{11} + \frac{\partial_\varphi D_{11}}{\sin \theta} + \frac{\partial_\psi D_{12}}{\cos \theta} \right], \\
U_1^\varphi \sin \theta &= \frac{2}{5} \left[D_{11} \left(\frac{2}{\tan \theta} - \tan \theta \right) + \partial_\theta D_{11} + \frac{\partial_\varphi C_{12}}{\sin \theta} + \frac{\partial_\psi D_{13}}{\cos \theta} \right], \\
U_1^\psi \cos \theta &= \frac{2}{5} \left[D_{12} \left(\frac{1}{\tan \theta} - 2 \tan \theta \right) + \partial_\theta D_{12} + \frac{\partial_\varphi D_{13}}{\sin \theta} - \frac{\partial_\psi (C_{11} + C_{12})}{\cos \theta} \right], \\
h_{AB} &= \bar{h}_{AB} + \frac{q_{AB}}{r^{3/2}} + \mathcal{O}(r^{-5/2}) \quad \text{with} \quad \bar{h}_{AB} = \text{diag}\{1, \sin^2 \theta, \cos^2 \theta\}, \tag{A.3}
\end{aligned}$$

and

$$\begin{aligned}
q_{\theta\theta} &= C_{11} \quad , \quad q_{\theta\varphi} = \sin \theta D_{11} \quad , \quad q_{\theta\psi} = \cos \theta D_{12}, \\
q_{\varphi\varphi} &= \sin^2 \theta C_{12} \quad , \quad q_{\varphi\psi} = \sin \theta \cos \theta D_{13} \quad , \quad q_{\psi\psi} = \cos^2 \theta C_{13}; \tag{A.4}
\end{aligned}$$

The three **C**'s and three **D**'s are functions of $\{u, x^A\}$, they are not independent:

$$C_{11} + C_{21} + C_{31} = 0 \quad , \quad C_{12} + C_{22} + C_{32} = 0 \quad , \quad C_{13} + C_{23} + C_{33} = D_{11}^2 + D_{21}^2 + D_{31}^2. \tag{A.5}$$

There are thus 5 independent functions associated with the independent degrees of freedom of the gravitational waves in five dimensions. $\mathcal{E}(u, x^A)$ is an integration ‘‘constant’’.

The background at infinity is flat. Its line element in our coordinates is

$$\overline{ds}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -du^2 - 2dudr + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2) \Rightarrow \sqrt{-\bar{g}} = \frac{1}{2} r^3 \sin 2\theta. \tag{A.6}$$

To calculate the mass loss we shall need the inverse components of the metric $g^{\mu\nu}$; the non-zero ones are as follows:

$$\begin{aligned}
g^{01} &= -1 + \frac{B_1}{r^3} + \mathcal{O}(r^{-7/2}) \quad , \quad g^{11} = 1 + \frac{V_1}{r^{3/2}} - \frac{\mathcal{E}}{r^2} - \frac{B_1}{r^3} + \mathcal{O}(r^{-7/2}) \quad , \quad g^{1A} = \frac{U_1^A}{r^{5/2}} + \frac{U_2^A}{r^{7/2}} + \mathcal{O}(r^{-11/2}), \\
g^{22} &= \frac{1}{r^2} \left(1 - \frac{C_{11}}{r^{3/2}} - \frac{C_{12}}{r^{5/2}} \right) + \frac{1}{r^5} \left(\frac{1}{2} C_{11}^2 - C_{13} + D_{11}^2 + D_{12}^2 \right) + \mathcal{O}(r^{-11/2}), \\
g^{23} &= -\frac{1}{r^3 \sin \theta} \left(\frac{D_{11}}{r^{1/2}} + \frac{D_{12}}{r^{3/2}} \right) + \frac{1}{r^5 \sin \theta} [-C_{13} D_{11} + (D_{12} - 1) D_{13}] + \mathcal{O}(r^{-11/2}), \\
g^{24} &= -\frac{1}{r^3 \sin \theta} \left(\frac{D_{12}}{r^{1/2}} + \frac{D_{22}}{r^{3/2}} \right) + \frac{1}{r^5 \sin \theta} (-C_{12} D_{12} - D_{23} + D_{11} D_{13}) + \mathcal{O}(r^{-11/2}), \\
g^{33} &= \frac{1}{r^2 \sin^2 \theta} \left(1 - \frac{C_{12}}{r^{3/2}} - \frac{C_{22}}{r^{5/2}} \right) + \frac{1}{r^5 \sin^2 \theta} \left(\frac{1}{2} C_{12}^2 - C_{23} + D_{11}^2 + D_{13}^2 \right) + \mathcal{O}(r^{-11/2})
\end{aligned}$$

$$g^{34} = -\frac{2}{r^2 \sin 2\theta} \left(\frac{D_{13}}{r^{3/2}} + \frac{D_{23}}{r^{5/2}} \right) + -\frac{2}{r^5 \sin 2\theta} (C_{11}D_{13} + D_{33} - D_{11}D_{12}) + \mathcal{O}(r^{-11/2}),$$

$$g^{44} = \frac{1}{r^2 \sin^2 \theta} \left(1 - \frac{C_{13}}{r^{3/2}} - \frac{C_{23}}{r^{5/2}} \right) + \frac{1}{r^5 \sin^2 \theta} \left(\frac{1}{2} C_{13}^2 - C_{33} + D_{12}^2 + D_{13}^2 \right) + \mathcal{O}(r^{-11/2}). \quad (\text{A.7})$$

$g^{\mu\nu}$ appears in the Christoffel symbols $\Gamma_{\mu\nu}^\lambda$ and in the end we need $\Delta_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \bar{\Gamma}_{\mu\nu}^\lambda$. The background calculations are not particularly difficult. One will not need all the Δ 's for the energy associated with the time translation Killing vector $\xi = \{1, 0, 0, 0, 0\}$. Moreover $\sqrt{-g} \propto r^3$ and most of the $\sqrt{-g}\mathbf{U}_1$'s go to zero when $r \rightarrow \infty$; in the end the only remaining term in $\hat{\mathbf{U}}_1$ that contributes to the energy is

$$\sqrt{-g}\Delta_{01}^1 = \frac{1}{2} \sin 2\theta \mathcal{E} + \text{non contributing parts.} \quad (\text{A.8})$$

From this follows that the charge

$$E = \oint_S \mathcal{E} dS_3 \quad (\text{A.9})$$

Finally from Einstein's equations $R_{uu} = 0$ and $R_{ux^A} = 0$ we obtain, like Tanabe, Tanahashi and Shiromizu in [33] the Bondi mass loss; with $E = Mc^2$

$$\frac{dM}{du} = -\frac{c^2}{G_5} \frac{1}{S_3} \oint_S \left\{ \frac{1}{3} \left[(\partial_u C_{11} + \frac{1}{2} \partial_u C_{21})^2 + \frac{3}{4} (\partial_u C_{21})^2 + (\partial_u D_{11})^2 + (\partial_u D_{12})^2 + (\partial_u D_{13})^2 \right] \right\} dS_3. \quad (\text{A.10})$$

References

- [1] Arnowitt R, Deser S and Misner C W 1959 Dynamical structure and definition of energy in General Relativity *Phys. Rev.* **116** 1322
- [2] Aros R, Contreras M, Olea R, Troncoso R and Zanelli J 2000 Conserved charges for gravity with locally anti-de Sitter asymptotics *Phys. Rev. Lett.* **84** 1647
- [3] Allemandi G, Francaviglia M and Raiteri M 2003 Charges and Energy in Chern-Simons Theories and Lovelock Gravity *Class. Quantum Grav.* **20** 5103 (*Preprint* arXiv:gr-qc/0308019)
- [4] Amery G and Shellard E P S 2003 Causal perturbation theory in general FRW cosmologies I: Energy-momentum conservation and matching conditions *Phys. Rev. D* **67** 083502 [astro-ph/0207146].

- [5] Bondi H 1960 Gravitational waves in general relativity *Nature* **186** 535
- [6] Deruelle N and Katz J 2005 On the Mass of a Kerr-anti-de Sitter Spacetime in D Dimensions *Class. Quantum Grav.* **22** 421 (*Preprint* arXiv:gr-qc/0410135 v1)
- [7] Deser S and Tekin B 2003 Energy in generic higher curvature gravity theories *Phys. Rev. D* **67** 084009 (*Preprint* hep-th/0212292)
- [8] Eisenhart L P *Non-Riemannian geometry* 1927 AMS Colloquium Publications **VIII** pp. 184. Reprinted in 2005 (New York: Dover Pub.)
- [9] Gibbons G W, Perry M J and Pope C N 2005 The First Law of Thermodynamics for Kerr-Anti-de Sitter Black Holes *Class. Quantum Grav.* **22** 1503 (*Preprint* arXiv:hep-th/0408217)
- [10] Goldberg J N 1963 Asymptotic Invariants in Gravitational Radiation Fields *Phys. Rev.* **131** 1367
- [11] Hollands S, Ishibashi S and Marolf D 2005 Counter-term charges generate bulk symmetries *Phys. Rev. D* **72** 104025 (*Preprint* arXiv:hep-th 0503105v3)
- [12] Henneaux M and Teitelboim C 1985 Asymptotically anti-de Sitter spaces *Commun. Math. Phys.* **424** 391
- [13] Iyer V and Wald R M 1994 Some properties of the Noether charge and a proposal for dynamical black hole entropy *Phys. Rev. D* **50** 846
- [14] Jasmin E 2008 A Note on Conserved Charges of Asymptotically Flat and Anti-de Sitter Spaces in Arbitrary Dimensions *Class. Quantum Grav.* **40** 2569 (*Preprint* arXiv:gr-qc/9804029 v2)
- [15] Jacobson T and Myers R C 1993 Entropy of Lovelock black holes *Phys. Rev. Lett.* **70** 3684 (*Preprint* arXiv:hep-th 09305016v1)
- [16] Julia B and Silva S 1998 Currents and superpotentials in classical gauge theories: I. Local results with applications to perfect fluids na General Relativity *Class. Quantum Grav.* **15** 2173 (*Preprint* ArXiv:0991 1035v2)
- [17] Julia B and Silva S 1999 On first order formulations of supergravity *JHEP* **0001** 026 (*Preprint* arXiv:0705.0484)

- [18] Julia B and Silva S 2000 Currents and superpotentials in classical gauge theories: II. Global aspects and the example of affine gravity *Class. Quantum Grav.* **17** 4733 (*Preprint* arXiv:0705.0484)
- [19] Julia B and Silva S 2002 On covariant phase space methods (*Preprint* arXiv:hep-th/0205.072)
- [20] Katz J 1985 A note on Komar's anomalous factor *Class. Quantum Grav.* **2** 423
- [21] Katz J and Lerer D 1997 On global conservation laws at null infinity *Class. Quantum Grav.* **14** 2249 (*Reprint* gr-qc/9612025)
- [22] Katz J and Livshits G I 2008 Superpotentials from variational derivatives rather than Lagrangians in relativistic theories of gravity *Class. Quantum Grav.* **25** 175024 (*Reprint* arXiv:0807.3079)
- [23] Katz J, Bičák J and Lynden-Bell D 1997 Relativistic conservation laws and integral constraints for large cosmological perturbations *Phys. Rev. D* **55** 5957 (*Reprint* gr-qc/0504041)
- [24] Kofinas G and Olea R 2007 Universal regularization for Lovelock AdS gravity *J. High Energy Phys.* **11** 069 (*Preprint* arXiv:0708.0782v2 [hep-th])
- [25] Komar A 1959 Covariant conservation laws in General Relativity *Phys. Rev.* **113**, 934
- [26] Lovelock D 1971 The Einstein Tensor and Its Generalizations *J. Math. Phys.* **12** 498
- [27] Mann R B, Marolf D and Virmani A 2006 Covariant counter-terms and conserved charges in asymptotically flat spacetimes *Class. Quantum Grav.* **23** 6357 (*Reprint* arXiv:gr-qc/0607041)
- [28] Obukhov Y N and Rubilar G F 2006 Covariance properties and regularization of conserved currents in tetrad gravity *Phys. Rev. D* **73** 124017 (*Reprint* arXiv:gr-qc/0605045Here is the Tex format for that reference)
- [29] Regge T and Teitelboim C 1974 Role of surface integrals in the Hamiltonian formulation of General Relativity *Ann. of Phys.* **88** 286
- [30] Rosen J 1974 Generalized Noether's Theorem. I. Theory *Ann. of Phys.* **82** 54

- [31] Sachs R K 1962 Gravitational waves in general relativity VIII. Waves in asymptotically flat spacetimes *Proc.Roy. Soc. A* **270** 103
- [32] Silva S 1999 On superpotentials and charge algebras of gauge theories *Nucl.Phys. B* **558** 391 (*Preprint* arXiv:hep-th/9809109)
- [33] Tanabe K, Tanahashi N and Shiromizu T 2010 On asymptotic structure at null infinity in five dimensions *J. Math. Phys.* **51** 062502 (*Reprint* arXiv:gr-qc/0909.0426v2)
- [34] Tanabe K, Tanahashi N and Shiromizu T 2011 Angular momentum at null infinity in five dimensions *J. Math. Phys.* **51** 062502 (*Reprint* arXiv:1010.1664v2 [gr-qc])
- [35] Von Freud P 1939 Über die Ausdrücke der Gesamtenergie und des Gesamtpulses eines Materiellen Systems in der Allgemeinen Relativitätstheorie *Ann. Math. J.* **40** 417
- [36] Wald R M and Zoupas A 2000 General definition of “conserved quantities” in General Relativity and other theories of gravity *Phys. Rev. D* **61**, 084027 (*Reprint* arXiv:gr-qc/9911095)
- [37] Zwiebach B 1985 Curvature squared terms and string theories *Phys. Letters B* **156** 315